

Simplified non-Navier-Stokes model of turbulent flow and its first numerical realization in D2

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General remarks on the simplified model

Let us consider the following integro-differential equation

$$(1) \quad \rho_t + \underline{\alpha} \nabla_{\underline{x}} \rho - \nu \Delta_{\underline{x}} \rho = \kappa \int_{\mathcal{A}} M(\cdot, \underline{\alpha}, \underline{\beta}) d\underline{\beta}$$

with properly chosen initial and boundary condition (of Dirichlet type). This is the equation of the simplified model proposed by Marek Burnat (see [1],[2]) with the additional diffusion term $-\nu \Delta_{\underline{x}} \rho$, where ν is a (small) positive coefficient.

Similarly as in papers [1] and [2], t , \underline{x} , $\underline{\alpha}$ and $\underline{\beta}$ are independent variables,

- $\rho(t, \underline{x}, \underline{\alpha})$ is the α -mass density function,
- $\underline{\alpha}, \underline{\beta}$ are α -velocities,
- the integral term $\kappa \int_{\mathcal{A}} M(\cdot, \underline{\alpha}, \underline{\beta})$ is the so called *mixer*.

We assume that

$$t \in [0, T], \quad T > 0, \quad \underline{x} \in \Omega, \quad \underline{\alpha}, \underline{\beta} \in \mathcal{A},$$

where Ω and \mathcal{A} are rectangles in \mathbf{R}^2 . Here we take

$$\Omega = [0, L_1] \times [0, L_2], \quad \mathcal{A} = [D_1, G_1] \times [D_2, G_2].$$

To get usual *Euler* quantities it is enough to integrate with respect to the variable $\underline{\alpha}$:

- for Euler mass density function

$$\varrho(t, \underline{x}) = \kappa \int_{\mathcal{A}} \rho(t, \underline{x}, \underline{\alpha}) d\underline{\alpha},$$

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- for Euler "impulse density function" (non official terminology!)

$$p(t, \underline{x}) = \kappa \int_{\mathcal{A}} \underline{\alpha} \rho(t, \underline{x}, \underline{\alpha}) d\underline{\alpha},$$

- for Euler impulse of the mass in $\omega \subset \Omega$

$$\text{imp}(t, \omega) = \int_{\omega} p(t, \omega) d\underline{x},$$

- for Euler velocity

$$\underline{v}(t, \underline{x}) = \frac{p(t, \underline{x})}{\varrho(t, \underline{x})} = \frac{\int_{\mathcal{A}} \underline{\alpha} \rho(t, \underline{x}, \underline{\alpha}) d\underline{\alpha}}{\int_{\mathcal{A}} \rho(t, \underline{x}, \underline{\alpha}) d\underline{\alpha}}.$$

We shall define the the function $M(\cdot, \underline{\alpha}, \underline{\beta})$ involved in the *mixer*. Let

$$d = \|\underline{\beta}\| \rho(t, \underline{x}, \underline{\beta}) - \|\underline{\alpha}\| \rho(t, \underline{x}, \underline{\alpha})$$

and

$$r(d) = \begin{cases} \frac{-d}{1+d} & \text{for } d \geq 0 \\ \frac{-d}{1-d} & \text{for } d < 0 \end{cases},$$

where $\|\cdot\|$ is the Euclidean norm. Then M is defined by

$$(2) \quad M(\cdot, \underline{\alpha}, \underline{\beta}) = \begin{cases} r(d) \rho(t, \underline{x}, \underline{\alpha}) & \text{if } d \geq 0 \\ r(d) \rho(t, \underline{x}, \underline{\beta}) & \text{if } d < 0 \end{cases}.$$

It is easy to verify that

$$\int_{\mathcal{A}} \int_{\mathcal{A}} M(\cdot, \underline{\alpha}, \underline{\beta}) d\underline{\beta} d\underline{\alpha} = 0.$$

Equation (1) is equivalent to

$$(3) \quad \kappa[\rho_t + \text{div}_{\underline{x}}(\underline{\alpha}\rho) - \nu \text{div}_{\underline{x}} \nabla_{\underline{x}} \rho] = \kappa^2 \int_{\mathcal{A}} M(\cdot, \underline{\alpha}, \underline{\beta}) d\underline{\beta}.$$

hence, integrating both sides of (3) with respect to $\underline{\alpha}$ we obtain the following equation for the *Euler quantities*:

$$(4) \quad \varrho(t, \underline{x}) + \text{div}_{\underline{x}} p(t, \underline{x}) - \nu \text{div}_{\underline{x}} \nabla_{\underline{x}} \varrho(t, \underline{x}) = 0,$$

Let us integrate both sides of (4) over $\omega \subset \Omega$. The Gauss Divergence Theorem implies the following relation for the mass $m(t, \omega) = \int_{\omega} \varrho(t, \underline{x}) d\underline{x}$ contained in ω :

$$(5) \quad m_t(t, \omega) + \int_{\partial\omega} \underline{n}p(t, \underline{x}) dS + \nu \int_{\partial\omega} \underline{n} \nabla_{\underline{x}} \varrho(t, \underline{x}) dS = 0,$$

where \underline{n} is the unit vector external normal to the boundary $\partial\omega$.

Observe that equation (5) can be read as the *the Mass Conservation Law* for the model considered.

Any change of the mass in ω is possible only as the result of fluxes through the boundary $\partial\omega$:

- of the impulse $\underline{n}p(t, \underline{x})$, and/or
- of the mass $\underline{n} \nabla_{\underline{x}} \varrho(t, \underline{x})$.

First numerical realization of the simplified model in 2D

Assume that rectangles Ω and \mathcal{A} are covered by grids Ω_h and \mathcal{A}_{ah} of steps h_i and ah_i $i = 1, 2$ respectively, where

$$(6) \quad \Omega_h = (x_{1k_1}, x_{2k_2}), \text{ and } \mathcal{A}_{ah} = (\alpha_{1l_1}, \alpha_{2l_2})$$

where $x_{ik_j} = h_i k_j$ and $\alpha_{il_j} = ah_i l_j$, $\underline{k} = (k_1, k_2)$, $\underline{l} = (l_1, l_2)$,

$$h_i = \frac{L_i}{M_i}, \quad ah_i = \frac{G_j - D_j}{PR_j - MR_j},$$

$$0 \leq k_i \leq M_i, \quad MR_j \leq l_j \leq PR_j, \quad i, j = 1, 2.$$

The grid for time interval $[0, T]$, $T > 0$, of the time-step $\tau = \frac{T}{N}$ is as follows

$$(7) \quad T_{\tau} = (t_n), \quad t_n = n\tau, \quad n = 0, 1, 2, \dots, N.$$

For approximation of the function ρ on the grid (6) (7) let us introduce the grid function

$$(8) \quad u_{\underline{k}, \underline{l}}^n = u_{k_1, k_2, l_1, l_2}^n \approx \rho(t_n, \underline{x}_{k_1, k_2}, \underline{\alpha}_{l_1, l_2})$$

where $\underline{x}_{k_1, k_2} = (h_1 k_1, h_2 k_2)$, $\underline{\alpha}_{l_1, l_2} = (ah_1 l_1, ah_2 l_2)$.

The grid function defined by (8) has to satisfy the following finite difference equation

$$(9) \quad \begin{aligned} du_{\underline{k}, \underline{l}}^{n+1} + a_1 u_{k_1-1, k_2, \underline{l}}^{n+1} + b_1 u_{k_1+1, k_2, \underline{l}}^{n+1} + a_2 u_{k_1, k_2-1, \underline{l}}^{n+1} + b_2 u_{k_1, k_2+1, \underline{l}}^{n+1} = \\ = d_1 u_{\underline{k}, \underline{l}}^n - a_1 u_{k_1-1, k_2, \underline{l}}^n - b_1 u_{k_1+1, k_2, \underline{l}}^n - a_2 u_{k_1, k_2-1, \underline{l}}^n - b_2 u_{k_1, k_2+1, \underline{l}}^n + \\ + \frac{\tau \kappa}{2} (\mathbf{F}(\underline{u}^{n+1}) + \mathbf{F}(\underline{u}^n)) + \mathbf{dir}_1 + \mathbf{dir}_2 \end{aligned}$$

with

$$\begin{aligned} a_1 &= -\lambda_1 \frac{ah_1 l_1}{4} - \frac{\nu \mu_1}{2}, \quad b_1 = -\lambda_1 \frac{ah_1 l_1}{4} + \frac{\nu \mu_1}{2}, \\ a_2 &= -\lambda_2 \frac{ah_2 l_2}{4} - \frac{\nu \mu_2}{2}, \quad b_2 = -\lambda_2 \frac{ah_2 l_2}{4} + \frac{\nu \mu_2}{2}, \\ d &= 1 + \nu(\mu_1 + \mu_2), \quad d_1 = 1 - \nu(\mu_1 + \mu_2) \\ \lambda_1 &= \frac{\tau}{h_1}, \quad \lambda_2 = \frac{\tau}{h_2}, \quad \mu_1 = \frac{\lambda_1}{h_1}, \quad \mu_2 = \frac{\lambda_2}{h_2}. \end{aligned}$$

The argument \underline{u}^n of the function \mathbf{F} is the long (block) vector

$$\underline{u}^n = [\underline{u}_{\underline{k}, \underline{l}_0}^n | \underline{u}_{\underline{k}, \underline{l}_1}^n | \cdots | \underline{u}_{\underline{k}, \underline{l}_q}^n]^T,$$

where $q = (MR_1 + PR_1)(MR_2 + PR_2) - 1$ ($q+1$ is the number of all pairs of indexes $\underline{l} = (l_1, l_2)$). Function $\mathbf{F}(\cdot)$ is the result of the trapezoidal quadrature in D2 of $M(\cdot, \cdot, \cdot)$ over \mathcal{A} with respect to the variable $\underline{\beta}$ (see definition of the mixer). Note that \mathbf{F} is a term non linear with respect to \underline{u} .

Dirichlet Boundary conditions are introduced by means of variables \mathbf{dir}_1 and \mathbf{dir}_2 :

$$\mathbf{dir}_1 = \begin{cases} -a_1(DIRL(n) + DIRL(n+1)) & \text{if } k_1 = 0 \\ -b_1(DIRR(n) + DIRR(n+1)) & \text{if } k_1 = M_1 - 1, \\ else & 0 \end{cases}$$

$$\mathbf{dir}_2 = \begin{cases} -a_2(DIRB(n) + DIRB(n+1)) & \text{if } k_2 = 0 \\ -b_2(DIRT(n) + DIRT(n+1)) & \text{if } k_2 = M_2 - 1 \\ else & 0 \end{cases}$$

Here $DIRL(\cdot)$, $DIRR(\cdot)$, $DIRB(\cdot)$, $DIRT(\cdot)$ are Dirichlet conditions at the left side, right side at the bottom and on the top side of the rectangle Ω respectively. Time level is given as argument of $DIR \cdot (\cdot)$.

Equation (9) is simply finite difference approximation of the equation (1) on the grid defined in (6)(7). Finite difference approximation is obtained as follows:

- first partial derivatives with respect to variables x_1 and x_2 : arithmetic mean of central finite differences at time levels n and $n+1$
- similarly, second partial derivatives with respect to x_1 and x_2 : arithmetic mean of second (forward-backward) finite differences at time levels n and $n+1$
- at both time levels n and $n+1$ corresponding Dirichlet boundary conditions are taken into account
- the mixer term is approximated using the trapezoidal 2D quadrature with respect to the variable β at time levels n and $n+1$; this gives corresponding nonlinear terms F . Finally, the arithmetic mean of levels n and $n+1$ is taken.

Let $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ be the matrices of dimension

$$(q+1)M_1M_2 \times (q+1)M_1M_2$$

corresponding to the left and right hand side of the linear part of equation (9), respectively. We can now write down a compact form of equations (9):

$$(10) \quad \underline{\mathbf{A}} \underline{u}^{n+1} = \underline{\mathbf{B}} \underline{u}^n + \frac{\tau\kappa}{2} [\mathbf{F}(\underline{u}^{n+1}) + \mathbf{F}(\underline{u}^n)] + \mathbf{DIR}$$

where \mathbf{DIR} is the sum of all terms introduced by Dirichlet boundary conditions. Matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ have a block-diagonal structure,

$$\underline{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & & & & \\ & \mathbf{A} & & & \\ & & \mathbf{A} & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \mathbf{A} \end{bmatrix}, \quad \underline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & & & & \\ & \mathbf{B} & & & \\ & & \mathbf{B} & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \mathbf{B} \end{bmatrix}$$

each diagonal block corresponds to part of system (9) that depends on a fixed \underline{l} only; hence the block dimension of the matrices $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ is equal to $q + 1 \times q + 1$. Matrices \mathbf{A} and \mathbf{B} of block dimension $M_2 \times M_2$ have also a block structure:

$$\mathbf{A} = \begin{bmatrix} D & B & & & \\ A & D & B & & \\ & A & D & B & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & A & D \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} D_1 & -B & & & \\ -A & D_1 & -B & & \\ & -A & D_1 & -B & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & -B & D_1 \end{bmatrix}$$

where A, B, D, D_1 are of dimension $M_1 \times M_1$ and

$$D = \begin{bmatrix} d & b_1 & & & \\ a_1 & d & b_1 & & \\ & a_1 & d & b_1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & a_1 & d \end{bmatrix}, \quad D_1 = \begin{bmatrix} d_1 & -b_1 & & & \\ -a_1 & d_1 & -b_1 & & \\ & -a_1 & d_1 & -b_1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & -a_1 & d_1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_2 & & & & \\ & a_2 & & & \\ & & a_2 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & a_2 & \end{bmatrix}, \quad B = \begin{bmatrix} b_2 & & & & \\ & b_2 & & & \\ & & b_2 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & b_2 \end{bmatrix}$$

Solving system (10)

As is proved in [3], the mixer function M is Lipschitz continuous with respect to the variable ρ . Hence it follows that function \mathbf{F} is Lipschitz continuous as well. This fact is very important for method of solving the nonlinear system (10). In order to solve approximately the system (10), one can apply the following simple iteration:

$$(11) \quad \underline{\mathbf{A}} \underline{u}^{(p+1)} = \underline{\mathbf{B}} \underline{u}^n + \frac{\tau \kappa}{2} [\mathbf{F}(\underline{\mathbf{u}}^{(p)}) + \mathbf{F}(\underline{\mathbf{u}}^n)] + \mathbf{DIR},$$

where p is the iteration index. Iteration converges for τ small enough. At any time step n we can put $\underline{u}^{(0)} = \underline{u}^n$ and if $\|\underline{u}^{(p+1)} - \underline{u}^{(p)}\|$ is not too large, then we accept $\underline{u}^{n+1} \approx \underline{u}^{(p+1)}$. At each iteration step it is necessary to solve linear system with the (non-symmetric) matrix $\underline{\mathbf{A}}$:

$$(12) \quad \underline{\mathbf{A}} x = \mathbf{f},$$

where \mathbf{f} is the sum of all known terms in (11), i.e. terms independent of $p+1$. To this end the *Richardson iteration* was applied:

$$(13) \quad x_{k+1} = x_k + sr_k, \quad r_k = \mathbf{f} - \underline{\mathbf{A}}x_k$$

with optimally chosen relaxation coefficient s . Let us estimate the optimal relaxation coefficient s corresponding to maximum norm $\|[x_1, x_2, \dots, x_N]^T\|_\infty = \max_{j=1,2,\dots,N} |x_j|$. If x is the solution of linear system (12), then for errors e_k and e_{k+1} we have

$$e_{k+1} = (I - s\underline{\mathbf{A}})e_k.$$

Hence

$$\|e_{k+1}\|_\infty \leq \|I - s\underline{\mathbf{A}}\|_\infty \|e_k\|_\infty.$$

Assume

$$\underline{\mathbf{A}} = \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdot & a_{1,N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{N,0} & a_{N,1} & a_{N,2} & \cdot & a_{N,N} \end{bmatrix}.$$

and $a_{i,i} \geq 0$. Then it is easy to verify that

$$(14) \quad \|I - s\underline{\mathbf{A}}\|_\infty = \max_{0 \leq i \leq N} (|1 - sa_{i,i}| + \sum_{j \neq i} |a_{i,j}|).$$

If $\sum_{j \neq i} |a_{i,j}| \leq a_{i,i}$ for all $0 \leq i \leq N$, then $\sum_{j \neq i} |a_{i,j}| + |1 - sa_{i,i}|$ takes minimal value at $s_{\text{opt}} = \frac{1}{a_{i,i}}$. In the case discussed $a_{i,i} = d > 0$ and

$$\sum_{j \neq i} |a_{i,j}| \leq |a_1| + |a_2| + |b_1| + |b_2|,$$

hence

$$\|I - s_{\text{opt}}\underline{\mathbf{A}}\|_\infty \leq \frac{|a_1| + |a_2| + |b_1| + |b_2|}{d} = \mathbf{NOR}.$$

Looking at the coefficients of equation (9), we infer that the sufficient condition for convergence of the Richardson iteration, $\mathbf{NOR} < 1$ is satisfied if proportions of steps in grids (6)(7) are properly chosen, i.e. if $\lambda_i = \frac{\tau}{h_i}$ and $\mu_i = \frac{\lambda_i}{h_i}$ for $i = 1, 2$ are small enough.

Computational experiment "Collision"

This experiment was run on the cluster halo2 of the Interdisciplinary Center for Mathematical and Computational Modeling of the University of Warsaw.

Experiment have to be considered as fully "virtual", because coefficients of the model have been taken more or less arbitrarily. Till now we had no possibility to confront our computational experiments with reality. In such situation, the results obtained have only a qualitative character.

Description of the experiment.

Four streams of mass enter into the "empty" rectangle Ω through its four sides. The entering streams are modeled by the Dirichlet boundary conditions for the function ρ , defined on each of four sides of Ω by positive functions with graphs of triangular shapes and of height increasing in time. Whole experiment contains **1000 time steps**. After **120 time steps** the four streams meet in the center of Ω and the first period of stagnation begins. But the mass is always entering into Ω and some parts of the streams start to go back, rubbing against parts of streams moving in opposite directions. After **520 time steps** the first period of stagnation ends, and first eddies appear. This period of turbulence ends after **610 time steps**, and in this moment the second period of stagnation begin, which ends after **870 time steps**. At this point the second period of turbulence starts with new eddies appearing.

Figures below give certain more interesting stages of this process. The field of unit Euler velocity vectors on the central part of Ω can be seen. In the preprint version of this paper (see < www.mimuw.edu.pl/preprints >) one can find more illustrations concerning the experiment "Collision".

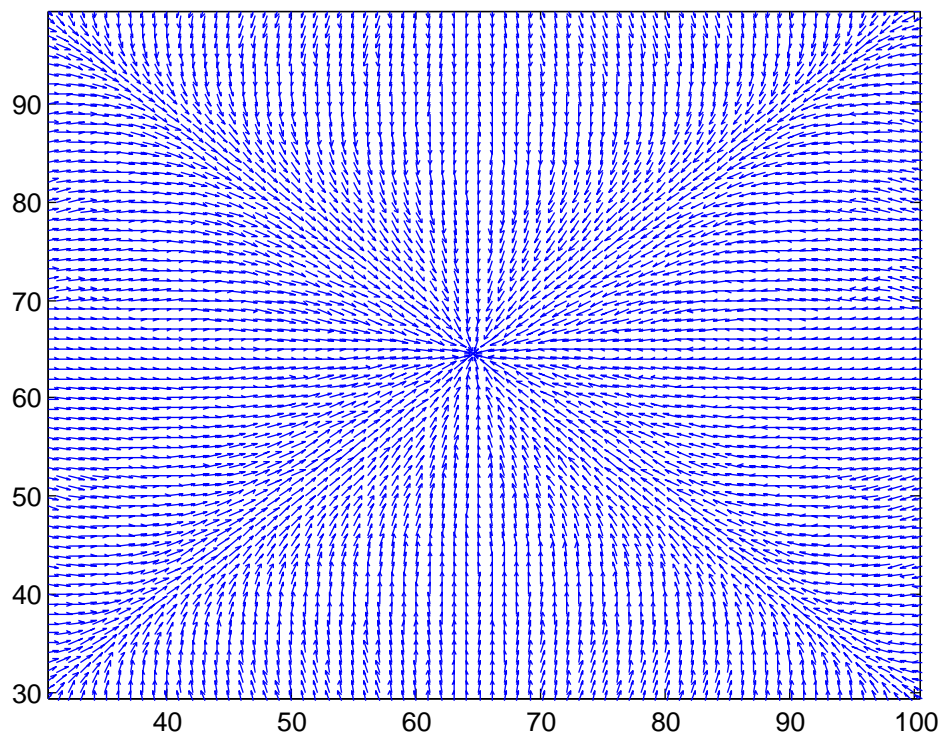
About the program

The program that realizes the algorithm described above, was build and can be run on the cluster halo2 of the Interdisciplinary Center for Mathematical and Computational Modeling of the University of Warsaw. Admissible dimension of the problem depends on the number of processors used. Coefficients, initial and boundary conditions, as well as the the number of processors can be chosen by the user.

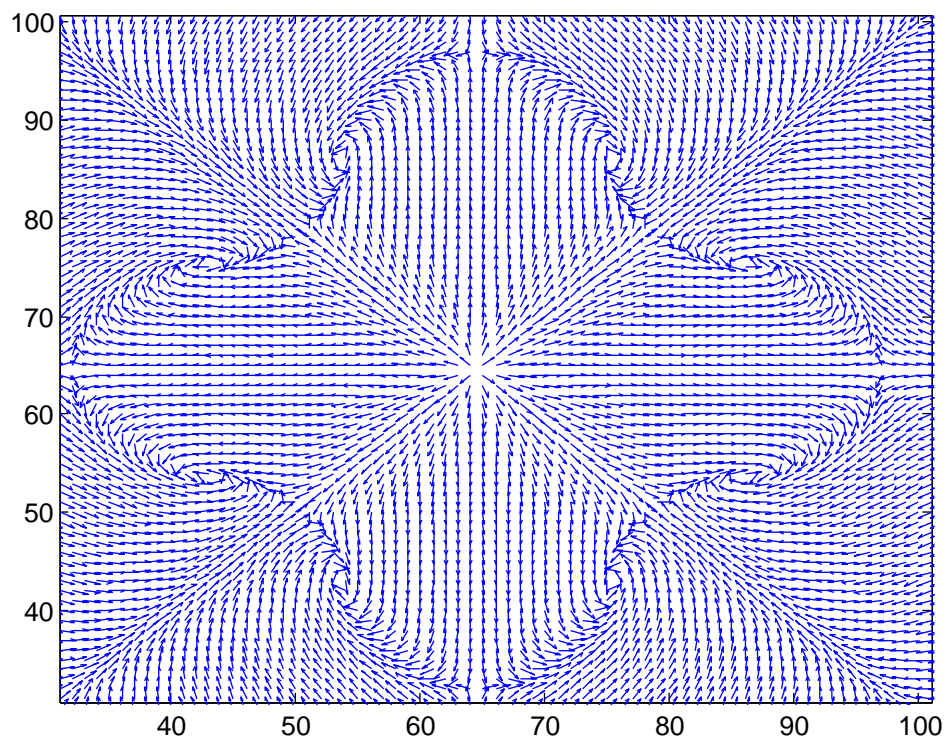
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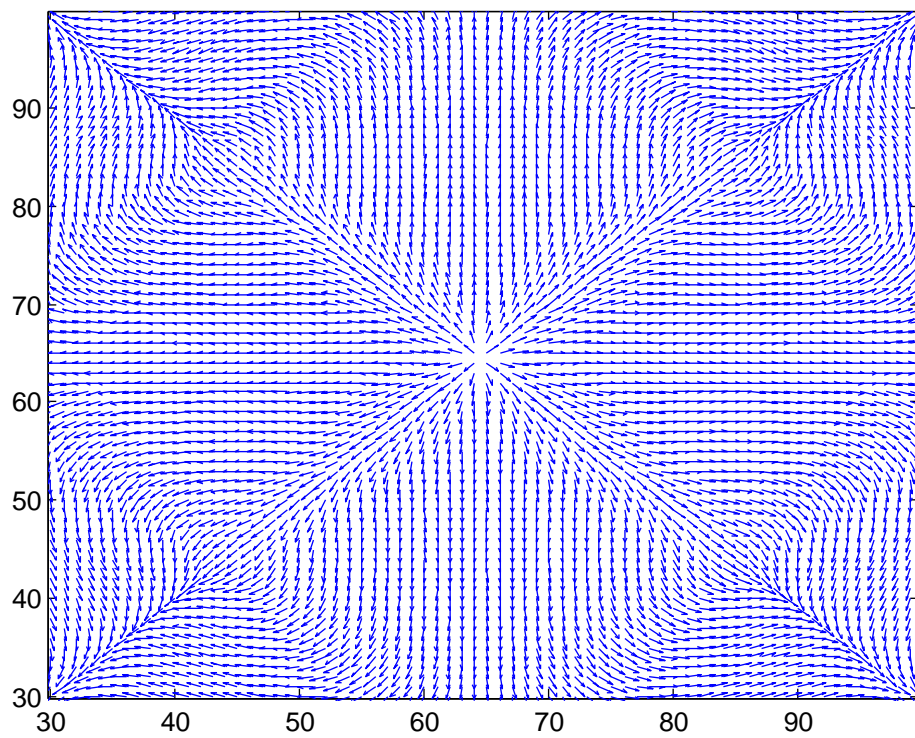
AFTER 500 TIME STEPS



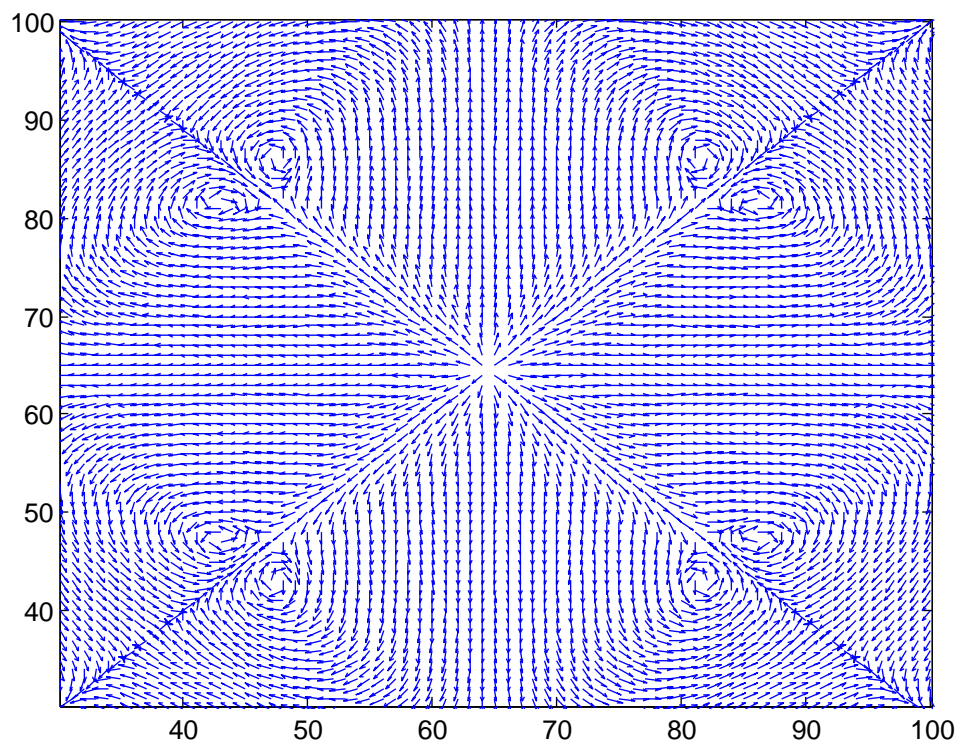
AFTER 560 TIME STEPS



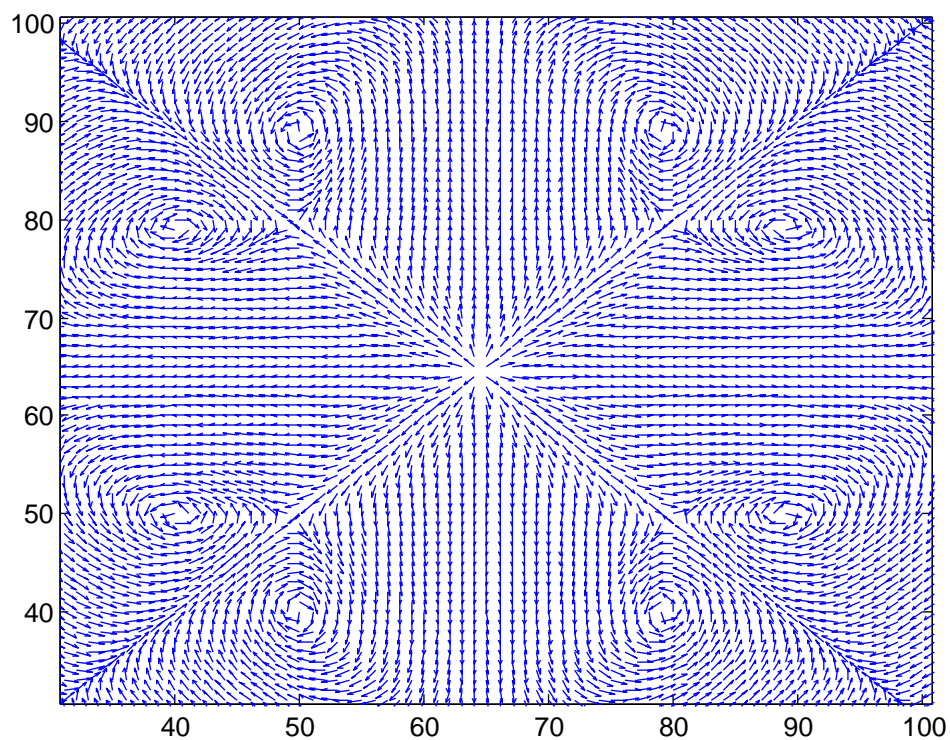
AFTER 850 TIME STEPS



AFTER 880 TIME STEPS



AFTER 1000 TIME STEPS



AFTER 1000 TIME STEPS (MAGNIFICATION OF A PART)

